Kuwait University | Math 102 | May 13, 2010
Department of Mathematics | Second Exam | Duration: 90 minutes

Calculators and mobile phones are not allowed.
Answer all of the following questions.

1. Evaluate the following. [3.5 pts. each]

(a) $\displaystyle\int \frac{2x}{(x-1)^2(x^2+1)}\,dx$

(b) $\displaystyle\int \frac{x}{1+\sqrt{2+x}}\,dx$

(c) $\displaystyle\int \frac{x^3}{\sqrt{x^2-1}}\,dx$

(d) $\displaystyle\int x^2 \tan^{-1} x\,dx$

(e) $\displaystyle\int \frac{1}{5-3\cos x}\,dx$

(f) $\displaystyle\int x\sqrt{2x-x^2}\,dx$

2. Determine whether the following improper integral is convergent or divergent. If it is convergent, find its value. [2 pts. each]

(a) $\displaystyle\int_0^1 \frac{dx}{(x+1)\sqrt{x}}$

(b) $\displaystyle\int_1^\infty \frac{dx}{(x+1)\sqrt{x}}$

Calculus (2): Midterm 2 - Solution2

Question 1: **(a)** $I = \int \frac{2x}{(x-1)^2(x^2+1)}dx = \int\left(\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}\right)dx$, where

$A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2 = 2x, \forall x \in R$. Let $x = 1 \to 2B = 2 \to \boxed{B = 1}$

$\to (A+C)x^3 + (-A+1+D-2C)x^2 + (A-2D+C)x + (-A+1+D) = 2x, \forall x \in R. \to$

$$\begin{cases} A+C=0 \\ -A+1+D-2C=0 \\ A-2D+C=2 \\ -A+1+D=0 \end{cases} \to \boxed{A=0, C=0, D=-1} \to I = \int\left(\frac{1}{(x-1)^2} - \frac{1}{x^2+1}\right)dx = \boxed{\frac{-1}{x-1} - \tan^{-1}x + c}$$

(b)

$$I = \int \frac{x}{1+\sqrt{2+x}}dx \overset{(\text{Let } u=\sqrt{2+x})}{=} 2\int \frac{u^3-2u}{1+u}du \overset{(\text{Long Division})}{=} 2\int\left(u^2 - u - 1 + \frac{1}{1+u}\right)du$$

$$= 2\left(\frac{u^3}{3} - \frac{u^2}{2} - u + \ln|1+u|\right) + c = \frac{2(2+x)^{3/2}}{3} - (2+x) - 2\sqrt{2+x} + 2\ln\left(1+\sqrt{2+x}\right) + c$$

(c) Method 1: $I = \int \frac{x^3}{\sqrt{x^2-1}}dx \overset{(\text{Let } u=\sqrt{x^2-1})}{=} \int(u^2+1)du = \frac{u^3}{3} + u + c = \frac{(x^2-1)^{3/2}}{3} + \sqrt{x^2-1} + c.$

Method 2: Let $x = \sec\theta \ (\theta = \sec^{-1}x) \to$

$$I = \int \sec^4\theta d\theta = \int(1+\tan^2\theta)\sec^2\theta d\theta = \frac{\tan^3\theta}{3} + \tan\theta + c$$

$$= \frac{\tan^3(\sec^{-1}x)}{3} + \tan(\sec^{-1}x) + c = \frac{(x^2-1)^{3/2}}{3} + \sqrt{x^2-1} + c.$$

(d)

$$I = \int x^2\tan^{-1}x dx \overset{(\text{Integration by Parts})}{=} \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\int\frac{x^3}{1+x^2}dx \overset{(\text{Long Division})}{=} \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\int\left(x - \frac{x}{1+x^2}\right)dx$$

$$= \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\left(\frac{x^2}{2} - \frac{1}{2}\ln(1+x^2)\right) + c$$

(e)

$$I = \int \frac{1}{5-3\cos x}dx \overset{(\text{Let } u=\tan(x/2))}{=} \int \frac{1}{\frac{5+5u^2-3+3u^2}{1+u^2}}\frac{2}{1+u^2}du$$

$$= \int \frac{1}{4u^2+1}du = \frac{1}{2}\tan^{-1}(2u) + c = \frac{1}{2}\tan^{-1}(2\tan(x/2)) + c$$

(g) $I = \int x\sqrt{2x-x^2}\,dx \overset{(\text{Complete the Square})}{=} \int x\sqrt{1-(x-1)^2}\,dx$. Let $x-1 = \sin\theta \ (\theta = \sin^{-1}(x-1)) \to$

$$I = \int(1+\sin\theta)\cos^2\theta d\theta = \int(\cos^2\theta + \sin\theta\cos^2\theta)d\theta = \int\left(\frac{1+\cos(2\theta)}{2} + \sin\theta\cos^2\theta\right)d\theta = \frac{\theta}{2} + \frac{\sin(2\theta)}{4} - \frac{\cos^3\theta}{3} + c$$

$$= \frac{\theta}{2} + \frac{\sin\theta\cos\theta}{2} - \frac{\cos^3\theta}{3} + c \overset{(\theta=\sin^{-1}((x-1)))}{=} \frac{1}{2}\sin^{-1}(x-1) + \frac{1}{2}(x-1)\sqrt{1-(x-1)^2} - \frac{1}{3}(1-(x-1)^2)^{3/2} + c$$

Qustion 2: $\int \frac{dx}{(x+1)\sqrt{x}} \overset{(\text{Let } u=\sqrt{x})}{=} \int \frac{2}{u^2+1}du = 2\tan^{-1}u + c = 2\tan^{-1}\sqrt{x} + c \to$

(a)

$\int_0^1 \frac{dx}{(x+1)\sqrt{x}} = \lim_{t\to0^+}\int_t^1 \frac{dx}{(x+1)\sqrt{x}} = \lim_{t\to0^+}[2\tan^{-1}\sqrt{x}]_t^1 = \lim_{t\to0^+}[2\tan^{-1}1 - 2\tan^{-1}\sqrt{t}] = \frac{\pi}{2} - 0 = \frac{\pi}{2}$ →Convergent

(b)

$\int_1^\infty \frac{dx}{(x+1)\sqrt{x}} = \lim_{t\to\infty}\int_1^\infty \frac{dx}{(x+1)\sqrt{x}} = \lim_{t\to\infty}[2\tan^{-1}\sqrt{x}]_1^t = \lim_{t\to\infty}[2\tan^{-1}\sqrt{t} - 2\tan^{-1}1] = \pi - \frac{\pi}{2} = \frac{\pi}{2}$ →Convergent